MTAT.05.118 Quantum Computing I
Homework # 4 + 5

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Handed out:	Monday, March 9
Ask questions:	Wednesday, March 11,18 (in class)
Due:	Monday March 16,23, by 12:15
	(in class on paper or as PDF by email to `rafieh.mosaheb@ut.ee`)

HW #4: Yes-Cloning Algorithm

In class, you heard about the no-cloning theorem. It's wrong. Here's a quantum algorithm which clones a register.

Input : n
register q of size n
Output: additional, new register r of size n

Prepare all qubits in reg. r in state $|0\rangle$
for $j = 0, \dots, n-1$ **do**
  Apply CNOT w/ control $q[j]$ and target $r[j]$

(a) Prove that for all $x \in \{0,1\}^n$, if the quantum circuit described by the algorithm is run with input n, $|x\rangle$, then the result is (input $\otimes$ output registers):

$$|x\rangle \otimes |x\rangle$$

(b) Hence, the no-cloning "theorem" is refuted. Find the mistake either in this algorithm / argument or in the proof of the no-cloning theorem.

HW #5: C$^{(n)}$X

(a) Using X (NOT), CX (CNOT) and CCX (Toffoli) gates, give a construction of a $C^{(n)}X$ "gate" on $n+1$ qubits. Your implementation may use a w-qubit "working memory" register — please indicate what w should be for your implementation. Denoting the qubit Hilbert space by $\mathbb{H}_2$, your "gate" is a unitary operator on $\mathbb{H}_2^{\otimes n} \otimes \mathbb{H}_2 \otimes \mathbb{H}_2^{w}$, whose effect on computational basis states is this:

$$C^{(n)} |x_1, \dots, x_n\rangle \otimes |y\rangle \otimes \left|0^{(w)}\right\rangle = |x_1, \dots, x_n\rangle \otimes \left|y \oplus \bigodot_{j=1}^{n} x_j\right\rangle \otimes \left|0^{(w)}\right\rangle$$

(where $\otimes$ denotes multiplication of Boolean numbers). In particular, you don't need to worry about the working memory being not in $|0\dots0\rangle$ state.

(b) Suppose you made the following mistake in the implementation of the unitary in (a): There exists at least one $x^{(b)} \in \{0,1\}^n$, $y^{(b)} \in \{0,1\}$ for which the working memory is not left in the state $\left|0^{(w)}\right\rangle$, but instead, the resulting state of the unitary is

$$|x_1^*, \ldots, x_n^*\rangle \otimes \left| y^* \oplus \bigodot_{j=1}^{n} x_j \right\rangle \otimes |psi\rangle$$

such that exists no θ for which $|\psi\rangle = e^{i\theta} \left|0^{(w)}\right\rangle$ (meaning: $|\psi\rangle$ is not $|0\rangle$ up to a phase). Also assume that the unitary works correctly for at least one $x^{(c)}, y^{(c)}$.
Give a quantum circuit with measurement operating only on the x- and y-registers (i.e., not using the working memory qubits, but taking auxilliary qubits is allowed, as always) which gives different measurement statistics in the broken and correct versions of the unitary. The numbers $x^{(b)}, y^{(b)}, x^{(c)}, y^{(c)}$ are assumed to be known.
In this sub-problem you are not restricted to X, CX, CCX, and you don't have to compile down to elementary gates.